

C^α regularity of weak solutions of non-homogenous ultraparabolic equations with drift terms

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April 19, 2017

Abstract

Consider a class of non-homogenous ultraparabolic differential equations with drift terms or lower order terms arising from some physical models, and we prove that weak solutions are Hölder continuous, which also generalizes the classic results of parabolic equations of second order. The main ingredients are a type of weak Poincaré inequality satisfied by non-negative weak sub-solutions and Moser iteration.

Keywords: Ultraparabolic equations, Moser iteration, Poincaré inequality, C^α regularity

1991 Mathematics Subject Classification. 35K70, 35H10, 35B65

1 Introduction

The ultraparabolic equations are of degenerate parabolic equations, which come from kinetic equations, diffusion process, Asian options and so on. One of the typical examples of ultraparabolic equations is the following equation of Kolmogorov type:

$$\frac{\partial u}{\partial t} + y \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial y^2} = 0, \quad (1.1)$$

which is of strong degenerated parabolic type equations.

Another example is arising from the Prandtl equations under Crocco transformation. The 2D Prandtl equations on $R_+^2 \times (0, T)$ (for example, see [28]) are as follows:

$$\begin{cases} \partial_t u + u \partial_x u + v \partial_y u + \partial_x \pi = \partial_{yy} u, & 0 < x < L, \quad y > 0, \\ \partial_x u + \partial_y v = 0, \\ u|_{t=0} = u_0(x, y), \quad u|_{y=0} = 0, \\ v|_{y=0} = v_0(x, t), \quad u|_{x=0} = u_1(y, t), \\ u(x, y, t) \rightarrow U(x, t), \quad y \rightarrow \infty, \end{cases} \quad (1.2)$$

where the pressure π is determined by the so-called Bernoulli's law:

$$\partial_t U + U \partial_x U + \partial_x \pi = 0.$$

From the physical background, one can assume that

$$U(x, t) > 0, \quad u_0(x, t) > 0, \quad u_1(y, t) > 0, \quad \text{and } v_0(x, t) \leq 0. \quad (1.3)$$

Under the monotone class assumptions

$$\partial_y u_0(x, t) > 0, \quad \partial_y u_1(y, t) > 0, \quad (1.4)$$

we use the following Crocco transformation:

$$\tau = t, \quad \xi = x, \quad \eta = \frac{u(x, y, t)}{U(x, t)}, \quad w(\tau, \xi, \eta) = \frac{\partial_y u(x, y, t)}{U(x, t)}.$$

Then the original Prandtl equations (1.2) is changed into

$$\partial_\tau w^{-1} + \eta U \partial_\xi w^{-1} + A \partial_\eta w^{-1} - B w^{-1} = -\partial_{\eta\eta} w \quad (1.5)$$

where $A = (1 - \eta^2) \partial_x U + (1 - \eta) \frac{\partial_t U}{U}$ and $B = \eta \partial_x U + \frac{\partial_t U}{U}$.

The authors in [28] proved that the equation (1.5) exists a global weak solution $w \in L^\infty(Q_T)$ with $Q_T = \{(\xi, \eta, \tau), 0 < \xi < L, 0 < \eta < 1, 0 < \tau < T\}$, if the pressure is favourable, i.e.

$$\partial_x \pi(x, t) \leq 0, \quad \forall 0 < x < L, t > 0. \quad (1.6)$$

It's still unknown whether the above weak solution is smooth. However, if the coefficient of (1.5) is smooth, the main part of the operator satisfies the well-known Hörmander's hypoellipticity conditions, which sheds lights on the smoothness of weak solutions. It is interesting whether weak solutions of equation (1.5) is still smooth when the coefficient is only measurable and local order terms exist.

On the other hand, recent results obtained by Pascucci-Polidoro [23] (see also, Cinti-Pascucci-Polidoro [5]) and Cinti-Polidoro [4]) proved that the Moser iterative method still works for a class of ultraparabolic equations with measurable coefficients. Their results show that for a non-negative sub-solution u of (1.1), the L^∞ norm of u is bounded by the L^p norm ($p \geq 1$). This is a very important step to the final regularity of solutions of the ultraparabolic equations. Based on these bounded estimates for weak solutions, Zhang [30], Wang-Zhang [25] and Wang-Zhang [26] can prove the Hölder regularity of weak solutions with the help of De Giorgi-Nash-Moser iteration by exploring a weak Poincaré inequality (see also, Xin-Zhang-Zhao [29], Wang-Zhang [27] for different ultraparabolic parabolic cases). Note that the above progress is based on the weak

solutions of (1.5) without lower order terms A and B . However, when lower order terms A or B exists, their integrability maybe change the relevant weak Sobolev and Poincaré inequalities and cause some difficulties in De Giorgi-Mash-Moser iterations.

In this paper, we are concerned with the C^α regularity of solutions of more general ultra-parabolic equations and consider the following non-homogeneous Kolmogorov-Fokker-Planck type operator on R^{N+1} :

$$Lu \equiv \sum_{i,j=1}^{m_0} \partial_i(a_{ij}(x,t)\partial_j u) + \sum_{i,j=1}^N b_{ij}x_i\partial_j u - \partial_t u = \sum_{i=1}^{m_0} b'_i(x,t)\partial_i u + c(x,t)u + f(x,t), \quad (1.7)$$

where $(x,t) \in R^{N+1}$, $1 \leq m_0 \leq N$, $\partial_{x_j} = \partial_j$ and b_{ij} is constant for every $i, j = 1, \dots, N$. Let $A = (a_{ij})_{N \times N}$, where $a_{ij} = 0$, if $i > m_0$ or $j > m_0$. Moreover, $b'(x,t) \in R^{m_0}$, $c(x,t), f(x,t) \in R$ are measurable functions. We make the following assumptions on the coefficients of L :

(H₁) $a_{ij} = a_{ji} \in L^\infty(R^{N+1})$ and there exists a $\lambda > 0$ such that

$$\frac{1}{\lambda} \sum_{i=1}^{m_0} \xi_i^2 \leq \sum_{i,j=1}^{m_0} a_{ij}(x,t) \xi_i \xi_j \leq \lambda \sum_{i=1}^{m_0} \xi_i^2$$

for every $(x,t) \in R^{N+1}$, and $\xi \in R^{m_0}$.

(H₂) The matrix $B = (b_{ij})_{N \times N}$ has the form

$$\begin{pmatrix} * & B_1 & 0 & \cdots & 0 \\ * & * & B_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & B_d \\ * & * & * & \cdots & * \end{pmatrix}$$

where B_k is a matrix $m_{k-1} \times m_k$ with rank m_k and $m_0 \geq m_1 \geq \dots \geq m_d$, $m_0 + m_1 + \dots + m_d = N$.

The requirements of matrix B in (H_2) ensure that the operator L with the constant a_{ij} satisfies the well-known Hörmander's hypoellipticity condition. We let $\lambda > 0$ and $\|B\| \leq \lambda$ in the sense of matrix norm. We refer to [5] for more details on non-homogeneous Kolmogorov-Fokker-Planck type operator on R^{N+1} .

The Schauder type estimate of (1.7) has been obtained for example, in [17, 18, 8]. Besides, the regularity of weak solutions have been studied by Bramanti-Cerutti-Manfredini [3], Polidoro-Ragusa [24] assuming a weak continuity on the coefficient a_{ij} . See also Angiuli-Lorenzi [1] for derivatives estimates by assuming the coefficients are Hölder continuous. It is quite interesting

whether the weak solution has Hölder regularity under the assumption (H_1) on a_{ij} . One of the approach to the Hölder estimates is to obtain the Harnack type inequality. In the case of elliptic equations with measurable coefficients, the Harnack inequality is obtained by Moser [20] via an estimate of BMO functions due to John-Nirenberg together with the Moser iteration method. Moser [21] also obtained the Harnack inequality for parabolic equations with measurable coefficients by generalizing the John-Nirenberg estimates to the parabolic case. Another approach to the Hölder estimates is given by Kruzhkov [13, 14] based on the Moser iteration to obtain a local priori estimates, which provides a short proof for the parabolic equations. Early De Giorgi [6] developed an approach to obtain the Hölder regularity for elliptic equations. Nash [22] also introduced another technique relying on the Poincaré inequality and obtained the Hölder regularity.

Our main line is to establish a type of weak Sobolev and Poincaré type inequalities for non-negative weak sub-solutions of (1.7). Then by using Kruzhkov's method of level sets we can obtain a local priori estimates which implies the Hölder estimates for ultraparabolic equation (1.7).

Next, we give a detailed definition of weak solution. Let D_{m_0} be the gradient with respect to the variables x_1, x_2, \dots, x_{m_0} . And

$$Y = \sum_{i,j=1}^N b_{ij} x_i \partial_{x_j} - \partial_t.$$

Definition 1.1 *We say that u is a weak solution of (1.7) in a domain $\Omega \subset R^{N+1}$ if it satisfies (1.7) in the distribution sense, that is for any $\phi \in C_0^\infty(\Omega)$, there holds*

$$\int_{\Omega} \phi Y u - (Du)^T A D \phi = \int_{\Omega} \phi (b' \cdot D_{m_0} u + cu + f), \quad (1.8)$$

and $u, D_{m_0} u, Y u, b', c, f \in L_{\text{loc}}^2(\Omega)$.

Similarly, we can define the *weak sub-solutions (super-solutions)* of (1.7) in a domain $\Omega \subset R^{N+1}$, if $u, D_{m_0} u, Y u, b', c, f \in L_{\text{loc}}^2(\Omega)$, and for any nonnegative $\phi \in C_0^\infty(\Omega)$, there holds

$$\int_{\Omega} \phi Y u - (Du)^T A D \phi \geq (\leq) \int_{\Omega} \phi (b' \cdot D_{m_0} u + cu + f). \quad (1.9)$$

One of the important feature of equation (1.7) is that the fundamental solution can be written explicitly if the coefficients a_{ij} is constant (cf. [15, 16]). Besides, there are some geometric and algebraic structures in the space R^{N+1} induced by the constant matrix B .

Recently in [11, 9], the authors consider the Laudau equation,

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (A[f] \nabla f + B[f] f)$$

and obtain Hölder continuity and Harnack inequality of weak solution by De Giorgi's method with bounded $A[f], B[f]$ by using an estimate of [2] for regularity gain in x direction. It is still unknown whether Harnack inequality holds in our general case (1.7). Another interesting application is regularity of weak solution of the Boltzman equation, for example see [12].

Now let us introduce some basic properties of hypoelliptic operator. Let $E(\tau) = \exp(-\tau B^T)$. For $(x, t), (\xi, \tau) \in R^{N+1}$, set

$$(x, t) \circ (\xi, \tau) = (\xi + E(\tau)x, t + \tau),$$

then (R^{N+1}, \circ) is a Lie group with identity element $(0, 0)$, and the inverse of an element is $(x, t)^{-1} = (-E(-t)x, -t)$. The left translation by (ξ, τ) given by

$$(x, t) \mapsto (\xi, \tau) \circ (x, t),$$

is a invariant translation to operator L when coefficient a_{ij} is constant. The associated dilation to operator L with constant coefficient a_{ij} is given by

$$\delta_t = \text{diag}(tI_{m_0}, t^3I_{m_1}, \dots, t^{2d+1}I_{m_d}, t^2),$$

where I_{m_k} denotes the $m_k \times m_k$ identity matrix, t is a positive parameter, also we assume

$$D_t = \text{diag}(tI_{m_0}, t^3I_{m_1}, \dots, t^{2d+1}I_{m_d}),$$

and denote

$$Q = m_0 + 3m_1 + \dots + (2d + 1)m_d,$$

then the number $Q + 2$ is usually called the homogeneous dimension of (R^{N+1}, \circ) with respect to the dilation δ_t .

(H₃): Let $c, f, b'(x, t)$ satisfy the conditions:

$$c, f \in L^q(\Omega) \text{ with } q > \frac{Q+2}{2}, \quad b' \in L^{Q+2}(\Omega). \quad (1.10)$$

Our main result is the following theorem.

Theorem 1.1 *Under the assumptions (H₁–H₃), weak solutions of (1.7) are Hölder continuous in Ω .*

Remark 1.1 *i) When $m_0 = N$, we have $Q = N$, and at this moment (1.7) becomes parabolic equations with general lower order terms. The requirements of coefficients in $(\mathbf{H}_1 - \mathbf{H}_3)$ agree with the sharp form of parabolic equations.*

ii) The above assumptions on b', c, f are due to the following embedding inequality. The conditions of $u \in L_t^\infty L_x^2$, $D_{m_0} u \in L_{loc}^2(\Omega)$, and the non-negative weak sub-solution u imply that $u \in L_{loc}^{\frac{2Q+4}{Q}}(\Omega)$.

Applying the above result to the equations (1.5), since there exists a global weak solution of (1.2) obtained in [28], we have the following conclusion.

Corollary 1.1 *[Interior regularity criterion for 2D Prandtl equation] Under the monotone class assumptions (1.3)-(1.4) and the favourable pressure (1.6), the global weak solution of (1.2) obtained in [28] is locally Hölder continuous if the conditions $u_t, U_t, U_x \in L_{loc}^6$ hold.*

The paper is organized as follows. In section 2, we introduce the structure of Lie group on the ultraparabolic operator and properties of fundamental solution. Section 3 is devoted to obtain some technical lemmas in proof of Theorem 1.1, including level set estimate with G-function method, weak Sobolev inequality, and weak Poincaré inequality. In section 4, we complete the proof of Theorem 1.1 and Corollary 1.1. The last section is an introduction to G-function.

2 Preliminary Results on Lie Groups

The norm in R^{N+1} , related to the group of translations and dilation to the equation is defined by

$$||(x, t)|| = r,$$

if r is the unique positive solution to the equation

$$\frac{x_1^2}{r^{2\alpha_1}} + \frac{x_2^2}{r^{2\alpha_2}} + \cdots + \frac{x_N^2}{r^{2\alpha_N}} + \frac{t^2}{r^4} = 1,$$

where $(x, t) \in R^{N+1} \setminus \{0\}$ and

$$\alpha_1 = \cdots = \alpha_{m_0} = 1, \quad \alpha_{m_0+1} = \cdots = \alpha_{m_0+m_1} = 3, \cdots,$$

$$\alpha_{m_0+\cdots+m_{d-1}+1} = \cdots = \alpha_N = 2d + 1.$$

And $||(0, 0)|| = 0$. The balls at a point (x_0, t_0) is defined by

$$\mathcal{B}_r(x_0, t_0) = \{(x, t) \mid ||(x_0, t_0)^{-1} \circ (x, t)|| \leq r\},$$

and

$$\mathcal{B}_r^-(x_0, t_0) = \mathcal{B}_r(x_0, t_0) \cap \{t < t_0\}.$$

For convenience, we sometimes use the cube replace the balls. The cube at point $(0, 0)$ is given by

$$\mathcal{C}_r(0, 0) = \{(x, t) \mid |t| \leq r^2, \quad |x_1| \leq r^{\alpha_1}, \dots, |x_N| \leq r^{\alpha_N}\}.$$

It is easy to see that there exists a constant Λ such that

$$\mathcal{C}_{\frac{r}{\Lambda}}(0, 0) \subset \mathcal{B}_r(0, 0) \subset \mathcal{C}_{\Lambda r}(0, 0),$$

where Λ only depends on N .

When the matrix $(a_{ij})_{N \times N}$ is constant matrix, we denoted it by A_0 , and A_0 has the form

$$A_0 = \begin{pmatrix} I_{m_0} & 0 \\ 0 & 0 \end{pmatrix}$$

then let

$$\mathcal{C}(t) \equiv \int_0^t E(s) A_0 E^T(s) ds,$$

which is positive when $t > 0$, and the operator L_1 takes the form

$$L_1 = \text{div}(A_0 D) + Y,$$

whose fundamental solution $\Gamma_1(\cdot, \zeta)$ with pole in $\zeta \in R^{N+1}$ has been constructed as follows:

$$\Gamma_1(z, \zeta) = \Gamma_1(\zeta^{-1} \circ z, 0), \quad z, \zeta \in R^{N+1}, \quad z \neq \zeta,$$

where $z = (x, t)$. And $\Gamma_1(z, 0)$ can be written down explicitly

$$\Gamma_1(z, 0) = \begin{cases} \frac{(4\pi)^{-\frac{N}{2}}}{\sqrt{\det \mathcal{C}(t)}} \exp(-\frac{1}{4} \langle \mathcal{C}^{-1}(t)x, x \rangle - t \text{tr}(B)) & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases} \quad (2.1)$$

There are some basic estimates for Γ_1 (see [2])

$$\Gamma_1(z, \zeta) \leq C \|\zeta^{-1} \circ z\|^{-Q}, \quad (2.2)$$

and

$$|\partial_{\xi_i} \Gamma_1(z, \zeta)| \leq C \|\zeta^{-1} \circ z\|^{-Q-1}, \quad (2.3)$$

where $i = 1, \dots, m_0$, for all $z, \zeta \in R^N \times (0, T]$.

Similarly, let $Y_0 = \langle x, B_0 D \rangle - \partial_t$, where B_0 has the form

$$\begin{pmatrix} 0 & B_1 & 0 & \cdots & 0 \\ 0 & 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & B_d \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

We denote $L_0 = \operatorname{div}(A_0 D) + Y_0$, and can define in the same way $E_0(t)$, $\mathcal{C}_0(t)$, and $\Gamma_0(z, \zeta)$ with respect to B_0 . We recall that $\mathcal{C}_0(t)$ ($t > 0$) (see [7]) satisfies

$$\mathcal{C}_0(t) = D_{t^{\frac{1}{2}}} \mathcal{C}_0(1) D_{t^{\frac{1}{2}}}. \quad (2.4)$$

The following lemma is obtained by Lanconelli and Polidoro (see [16]), which is need in our proof.

Lemma 2.1 *In addition to the above assumptions, for every given $T > 0$, there exist positive constants C_T and C'_T such that*

$$\langle \mathcal{C}_0(t)x, x \rangle (1 - C_T t) \leq \langle \mathcal{C}(t)x, x \rangle \leq \langle \mathcal{C}_0(t)x, x \rangle (1 + C_T t), \quad (2.5)$$

$$\langle \mathcal{C}_0^{-1}(t)x, x \rangle (1 - C_T t) \leq \langle \mathcal{C}^{-1}(t)x, x \rangle \leq \langle \mathcal{C}_0^{-1}(t)x, x \rangle (1 + C_T t), \quad (2.6)$$

$$C_T'^{-1} t^Q (1 - C_T t) \leq \det \mathcal{C}(t) \leq C'_T t^Q (1 + C_T t), \quad (2.7)$$

for every $(x, t) \in R^N \times (0, T]$ and $t < \frac{1}{C_T}$.

We copy a classical potential estimates (cf. (1.11) in [7]) here to prove the Poincaré type inequality.

Lemma 2.2 *Let (R^{N+1}, \circ) is a homogeneous Lie group of homogeneous dimension $Q + 2$, $\alpha \in (0, Q + 2)$ and $G \in C(R^{N+1} \setminus \{0\})$ be a δ_λ -homogeneous function of degree $\alpha - Q - 2$. If $f \in L^p(R^{N+1})$ for some $p \in (1, \infty)$, then*

$$G_f(z) \equiv \int_{R^{N+1}} G(\zeta^{-1} \circ z) f(\zeta) d\zeta,$$

is defined almost everywhere and there exists a constant $C = C(Q, p)$ such that

$$\|G_f\|_{L^q(R^{N+1})} \leq C \max_{\|z\|=1} |G(z)| \|f\|_{L^p(R^{N+1})},$$

where q is defined by

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q + 2}.$$

Corollary 2.1 *Let $f \in L^p(R^{N+1})$ with $1 < p < \infty$, recall the definitions in [5]*

$$\Gamma_1(f)(z) = \int_{R^{N+1}} \Gamma_1(z, \zeta) f(\zeta) d\zeta, \quad \forall z \in R^{N+1},$$

and

$$\Gamma_1(D_{m_0}f)(z) = - \int_{R^{N+1}} D_{m_0}^{(\zeta)} \Gamma_1(z, \zeta) f(\zeta) d\zeta, \quad \forall z \in R^{N+1},$$

then exists a positive constant $C = C(Q, T, B)$ such that

$$\|\Gamma_1(f)\|_{L^q(S_T)} \leq C \|f\|_{L^p(S_T)}, \quad \frac{1}{q} = \frac{1}{p} - \frac{2}{Q+2}$$

and

$$\|\Gamma_1(D_{m_0}f)\|_{L^q(S_T)} \leq C \|f\|_{L^p(S_T)}, \quad \frac{1}{q} = \frac{1}{p} - \frac{1}{Q+2}$$

3 Proof of Main Theorem

To obtain a local estimates of solutions of the equation (1.7), for instance, at point (x_0, t_0) , we may consider the estimates at a ball centered at $(0, 0)$, since the equation (1.7) is invariant under the left group translation when a_{ij} is constant. By introducing a type of weak Sobolev and Poincaré type inequality, we prove the following Lemma 3.8 which is essential in the oscillation estimates in Kruzhkov's approaches in parabolic case.

For convenience, let $x' = (x_1, \dots, x_{m_0})$ and $x = (x', \bar{x})$. Consider the estimates in the following cube, instead of \mathcal{B}_r^- ,

$$\mathcal{C}_r^- = \{(x, t) \mid -r^2 \leq t \leq 0, |x'| \leq r, |x_{m_0+1}| \leq (\lambda r)^3, \dots, |x_N| \leq (\lambda r)^{2d+1}\}.$$

Let

$$K_r = \{x' \mid |x'| \leq r\},$$

$$S_r = \{\bar{x} \mid |x_{m_0+1}| \leq (\lambda r)^3, \dots, |x_N| \leq (\lambda r)^{2d+1}\}.$$

Moreover, assume that $0 < \alpha, \beta < 1$ are constants, for fixed t and h , and

$$\mathcal{N}_{t,h} = \{(x', \bar{x}) \in K_{\beta r} \times S_{\beta r}, \quad u(\cdot, t) \geq h\}.$$

In the following discussions, we sometimes abuse the notations of \mathcal{B}_r^- and \mathcal{C}_r^- , since there are equivalent, and we always assume $r \ll 1$ and $\lambda > 8$ in the following arguments, since λ can choose a large constant. Moreover, all constants depend on m_0, d, N or Q will be denoted by dependence on B .

Lemma 3.1 Suppose that $u(x, t) \geq 0$ is a weak solution of equation (1.7) in Ω . Let $w = G(\gamma u + h)$ (see the definition of G in Lemma 5.1) with $\gamma > 0$ and $0 < h \leq \frac{1}{4}$. There holds the following inequality

$$\int_{\Omega} (Dw)^T AD(\eta^2) + \eta^2 (Dw)^T ADw - \eta^2 Yw \leq \int_{\Omega} [-\eta^2 b' \cdot D_{m_0} w + \eta^2 \frac{\gamma |cu + f|}{h}] \quad (3.1)$$

where $0 \leq \eta \in C_0^\infty(\Omega)$.

Proof. The argument follows from the properties of G-function as in [20, 14, 10], and more details we refer to Lemma 5.1 in the Appendix.

Let $w = G(\gamma u + h)$, then $D_{x_i} w = \gamma G'(\gamma u + h) D_{x_i} u$ for $i = 1, \dots, N$ and $D_t w = \gamma G'(\gamma u + h) D_t u$. Hence $\phi = \gamma G'(\gamma u + h) \eta^2$ can be a test function of the following integral equation

$$\int_{\Omega} \phi Y u - (Du)^T AD \phi = \int_{\Omega} \phi (b' \cdot D_{m_0} u + cu + f).$$

Thus, we get

$$\int_{\Omega} \eta^2 Y w - (Dw)^T AD(\eta^2) - \eta^2 \gamma^2 G''(\gamma u + h) (Du)^T AD u = \gamma \int_{\Omega} G'(\gamma u + h) \eta^2 (b' \cdot D_{m_0} u + cu + f).$$

Note that by Lemma 5.1, we know that

$$G''(\gamma u + h) \geq G'(\gamma u + h)^2, \quad |G'(\gamma u + h)| \leq \frac{1}{h},$$

and we derive that

$$\int_{\Omega} (Dw)^T AD(\eta^2) + \eta^2 (Dw)^T ADw - \eta^2 Yw \leq \int_{\Omega} -\eta^2 b' \cdot D_{m_0} w + \eta^2 \frac{\gamma |cu + f|}{h}.$$

◇

The following lemma of energy estimate is similar to that in [25].

Lemma 3.2 There exist constants $\alpha = \alpha(B)$, $\beta = \beta(B)$, $r_1 = r_1(\lambda, B) \leq 1$ and

$$h_1 = h_1(B, \lambda, \|b'\|_{L^{Q+2}(\mathcal{B}_1^-)}, \|c\|_{L^q(\mathcal{B}_1^-)}, \|f\|_{L^q(\mathcal{B}_1^-)}),$$

such that for any $h \leq h_1$ and $r^{2-\frac{Q+2}{q}} \leq \min\{r_1^{2-\frac{Q+2}{q}}, h^{\frac{9}{8}}\}$ we have the following conclusion. Suppose that $u(x, t) \geq 0$ is a solution of equation (1.7) in \mathcal{B}_r^- centered at $(0, 0)$ and

$$\text{mes}\{(x, t) \in \mathcal{B}_r^-, \quad u \geq 1\} \geq \frac{1}{2} \text{mes}(\mathcal{B}_r^-).$$

Then for almost all $t \in (-\alpha r^2, 0)$, we have

$$\text{mes}\{\mathcal{N}_{t,h}\} \geq \frac{1}{11} \text{mes}\{K_{\beta r} \times S_{\beta r}\}.$$

Proof: Let

$$v = G(u + h^{\frac{9}{8}}),$$

where h is a constant, $0 < h < \frac{1}{4}$, to be decided. Then by Lemma 3.1 v satisfies

$$\int_{\mathcal{B}_r^-} (Dv)^T AD\psi - \psi Yv + \psi (Dv)^T ADv \leq \int_{\mathcal{B}_r^-} \psi(-b' \cdot D_{m_0}v + \frac{|cu + f|}{h^{\frac{9}{8}}}),$$

where $0 \leq \psi \in C_0^\infty(\mathcal{B}_r^-)$.

Let $\eta(x')$ be a smooth cut-off function so that

$$\begin{cases} \eta(x') = 1, & \text{for } |x'| < \beta r, \\ \eta(x') = 0, & \text{for } |x'| \geq r. \end{cases}$$

Moreover, $0 \leq \eta \leq 1$ and $|D_{m_0}\eta| \leq \frac{2m_0}{(1-\beta)r}$.

Replacing $\eta^2(x')$ into the above inequality and integrating by parts on $K_r \times S_{\beta r} \times (\tau, t)$, we have

$$\begin{aligned} & \int_{K_{\beta r}} \int_{S_{\beta r}} v(t, x', \bar{x}) d\bar{x} dx' + \frac{1}{2\lambda} \int_{\tau}^t \int_{K_r} \int_{S_{\beta r}} \eta^2 |D_{m_0}v|^2 d\bar{x} dx' dt \\ & \leq \frac{C}{\beta^Q (1-\beta)^2} \text{mes}(S_{\beta r}) \text{mes}(K_{\beta r}) + \int_{\tau}^t \int_{K_r} \int_{S_{\beta r}} \eta^2 x^T B Dv d\bar{x} dx' dt \\ & \quad + \int_{K_r} \int_{S_{\beta r}} v(\tau, x', \bar{x}) d\bar{x} dx' + Cr^Q \|b'\|_{L^{Q+2}}^2 + Ch^{-\frac{9}{8}} r^Q r^{2-\frac{Q+2}{q}} \|g\|_q, \quad \text{a.e. } \tau, t \in (-r^2, 0), \end{aligned} \tag{3.2}$$

where $g = |cu + f|$, C only depends on λ and B . Let

$$I_B \equiv \int_{K_r} \int_{S_{\beta r}} \eta^2 \sum_{i,j=1}^N x_i b_{ij} \partial_{x_j} v d\bar{x} dx' = I_{B_1} + I_{B_2},$$

where

$$\begin{aligned} I_{B_1} &= \int_{K_r} \int_{S_{\beta r}} \eta^2 \sum_{i=1}^N \sum_{j=1}^{m_0} x_i b_{ij} \partial_{x_j} v d\bar{x} dx', \\ I_{B_2} &= \int_{K_r} \int_{S_{\beta r}} \eta^2 \sum_{i=1}^N \sum_{j=m_0+1}^N x_i b_{ij} \partial_{x_j} v d\bar{x} dx'. \end{aligned}$$

On the other hand

$$\begin{aligned} |I_{B_1}| &\leq \int_{K_r} \int_{S_{\beta r}} \varepsilon \eta^2 |D_{m_0}v|^2 + C_\varepsilon \eta^2 \sum_{j=1}^{m_0} \sum_{i=1}^N |x_i b_{ij}|^2 d\bar{x} dx' \\ &\leq \int_{K_r} \int_{S_{\beta r}} \varepsilon \eta^2 |D_{m_0}v|^2 d\bar{x} dx' + C(\varepsilon, B, \lambda) \beta^{-Q} |K_{\beta r}| |S_{\beta r}|, \end{aligned}$$

and

$$\begin{aligned}
|I_{B_2}| &\leq \left| \int_{K_r} \int_{S_{\beta r}} \eta^2 \sum_{i=1}^N \sum_{j=m_0+1}^N x_i b_{ij} \partial_{x_j} v d\bar{x} dx' \right| \\
&\leq \left| \int_{K_r} \int_{S_{\beta r}} -\eta^2 \sum_{i=1}^N \sum_{j>m_0} \delta_{ij} b_{ij} v d\bar{x} dx' \right| \\
&\quad + \left| \int_{K_r} \int_{\partial_j S_{\beta r}} \eta^2 \sum_{i=1}^N \sum_{j>m_0} x_i b_{ij} v d\bar{x}_j dx' \right| \\
&\leq \lambda N \beta^{-Q} |K_{\beta r}| |S_{\beta r}| \ln(h^{-\frac{9}{8}}) \\
&\quad + \lambda \sum_{i=1}^N \sum_{j>m_0} \frac{(\lambda r)^{\alpha_i}}{(\lambda r)^{\alpha_j}} \beta^{-2Q} |K_{\beta r}| |S_{\beta r}| \ln(h^{-\frac{9}{8}}),
\end{aligned}$$

where $\bar{x}_j = (x_{m_0+1}, \dots, x_{j-1}, x_{j+1}, \dots, x_N)$. When $\alpha_i \geq \alpha_j$, we have

$$\int_{\tau}^t |I_{B_2}| \leq (\lambda N r^2 + \lambda r^2 N^2) \beta^{-2Q} |K_{\beta r}| |S_{\beta r}| \ln(h^{-\frac{9}{8}}),$$

or $i < j$, thus $\alpha_j = \alpha_i + 2$ by the property of B , then

$$\int_{\tau}^t |I_{B_2}| \leq (\lambda N r^2 + \lambda^{-1} N^2) \beta^{-2Q} |K_{\beta r}| |S_{\beta r}| \ln(h^{-\frac{9}{8}}).$$

By $\lambda > 8$ choose r_1 small enough, such that for any $r \leq r_1$

$$\lambda N r^2 + \lambda r^2 N^2 + \lambda^{-1} N^2 < \frac{1}{8},$$

thus

$$\int_{\tau}^t |I_{B_2}| \leq \frac{1}{4} \beta^{-2Q} |K_{\beta r}| |S_{\beta r}| \ln(h^{-\frac{9}{8}}).$$

Integrating by t to I_B , we have

$$\begin{aligned}
&\int_{\tau}^t \int_{K_r} \int_{S_{\beta r}} \eta^2 x^T B D v d\bar{x} dx' dt \\
&\leq \frac{1}{4} \beta^{-2Q} \ln(h^{-\frac{9}{8}}) \text{mes}(S_{\beta r}) \text{mes}(K_{\beta r}) \\
&\quad + \int_{\tau}^t \int_{K_r} \int_{S_{\beta r}} \varepsilon \eta^2 |D_{m_0} v|^2 + C(\varepsilon, B, \lambda) \beta^{-Q} |K_{\beta r}| |S_{\beta r}|.
\end{aligned} \tag{3.3}$$

We shall estimate the measure of the set $\mathcal{N}_{t,h}$. Let

$$\mu(t) = \text{mes}\{(x', \bar{x}) \mid x' \in K_r, \bar{x} \in S_r, u(\cdot, t) \geq 1\}.$$

By our assumption, for $0 < \alpha < \frac{1}{2}$

$$\frac{1}{2} r^2 \text{mes}(S_r) \text{mes}(K_r) \leq \int_{-r^2}^0 \mu(t) dt = \int_{-r^2}^{-\alpha r^2} \mu(t) dt + \int_{-\alpha r^2}^0 \mu(t) dt,$$

that is

$$\int_{-r^2}^{-\alpha r^2} \mu(t) dt \geq \left(\frac{1}{2} - \alpha\right) r^2 \text{mes}(S_r) \text{mes}(K_r),$$

then there exists a $\tau \in (-r^2, -\alpha r^2)$, such that

$$\mu(\tau) \geq (\frac{1}{2} - \alpha)(1 - \alpha)^{-1} \text{mes}(S_r) \text{mes}(K_r),$$

we have by noticing $v = 0$ when $u \geq 1$,

$$\int_{K_r} \int_{S_{\beta r}} v(\tau, x', \bar{x}) d\bar{x} dx' \leq \frac{1}{2} (1 - \alpha)^{-1} \text{mes}(S_r) \text{mes}(K_r) \ln(h^{-\frac{9}{8}}).$$

Now we choose $\varepsilon = \frac{1}{2\lambda}$ and α (near zero) and β (near one), so that

$$\frac{1}{4\beta^{2Q}} + \frac{1}{2\beta^{2Q}(1 - \alpha)} \leq \frac{4}{5}. \quad (3.4)$$

Note that the last two terms of (3.2) and the last term in (3.3) can be controlled by

$$C(B, \lambda, \|b'\|_{L^{Q+2}}, \|c\|_{L^q}, \|f\|_{L^q})(1 - \beta)^{-2} \beta^{-Q} |K_{\beta r}| |S_{\beta r}|$$

by choosing $r^{2 - \frac{Q+2}{q}} \leq h^{\frac{9}{8}}$.

Combining (3.2)-(3.4), we deduce

$$\int_{K_{\beta r}} \int_{S_{\beta r}} v(t, x', \bar{x}) d\bar{x} dx' \leq [2C(1 - \beta)^{-2} \beta^{-Q} + \frac{4}{5} \ln(h^{-\frac{9}{8}})] \text{mes}(K_{\beta r} \times S_{\beta r}).$$

When $(x', \bar{x}) \notin \mathcal{N}_{t,h}$, $u \geq h$, we have

$$\ln(\frac{1}{2h}) \leq \ln^+(\frac{1}{h + h^{\frac{9}{8}}}) \leq v,$$

then

$$\ln(\frac{1}{2h}) \text{mes}(K_{\beta r} \times S_{\beta r} \setminus \mathcal{N}_{t,h}) \leq \int_{K_{\beta r}} \int_{S_{\beta r}} v(t, x', \bar{x}) d\bar{x} dx'.$$

Since

$$\frac{C + \frac{4}{5} \ln(h^{-\frac{9}{8}})}{\ln(h^{-1})} \longrightarrow \frac{9}{10}, \quad \text{as } h \rightarrow 0,$$

then there exists constant h_1 such that for $0 < h < h_1$ and $t \in (-\alpha r^2, 0)$

$$\text{mes}(K_{\beta r} \times S_{\beta r} \setminus \mathcal{N}_{t,h}) \leq \frac{10}{11} \text{mes}(K_{\beta r} \times S_{\beta r}).$$

Hence the proof is complete. \diamond

Let $\chi(s)$ be a smooth function given by

$$\begin{aligned} \chi(s) &= 1 & \text{if } s \leq \theta^{\frac{1}{Q}} r, \\ \chi(s) &= 0 & \text{if } s > r, \end{aligned}$$

where $\theta^{\frac{1}{Q}} < \frac{1}{2}$ is a constant. Moreover, we assume that

$$0 \leq -\chi'(s) \leq \frac{2}{(1 - \theta^{\frac{1}{Q}})r},$$

and $\chi'(s) < 0$, if $\theta^{\frac{1}{Q}}r < s < r$. Also for any β_1, β_2 , with $\theta^{\frac{1}{Q}} < \beta_1 < \beta_2 < 1$, we have

$$|\chi'(s)| \geq C(\beta_1, \beta_2) > 0,$$

if $\beta_1 r \leq s \leq \beta_2 r$.

For $x \in R^N$, $t < 0$, we set

$$\mathcal{Q} = \{(x', \bar{x}, t) \mid -r^2 \leq t < 0, x' \in K_{\frac{r}{\theta}}, |x_j| \leq \frac{r^{\alpha_j}}{\theta}, j = m_0 + 1, \dots, N\},$$

$$\phi_0(x, t) = \chi([\theta^2 \sum_{i=m_0+1}^N \frac{x_i^2}{r^{2\alpha_i-Q}} - C_1 t r^{Q-2}]^{\frac{1}{Q}}),$$

$$\phi_1(x, t) = \chi(\theta|x'|),$$

$$\phi(x, t) = \phi_0(x, t)\phi_1(x, t), \tag{3.5}$$

where $C_1 > 1$ is chosen so that

$$C_1 r^{Q-2} \geq \theta^2 |\sum_{i=1}^N \sum_{j>m_0} 2x_i b_{ij} x_j r^{Q-2\alpha_j}|,$$

for all $z \in \mathcal{Q}$.

In the following discussion, $a \approx b$ means

$$C(B, \lambda)^{-1}a \leq b \leq C(B, \lambda)a.$$

Remark 3.1 (c.f. Remark 3.1 in [25]) *By the definition of ϕ and the above arguments, it is easy to check that, for θ, r small and $t \leq 0$*

- (1) $\phi(z) \equiv 1$, in $\mathcal{B}_{\theta r}^-$,
- (2) $\text{supp}\phi \cap \{(x, t); t \leq 0\} \subset \mathcal{Q}$,
- (3) *there exists $\alpha_1 > 0$, which depends on C_1 , such that*

$$\{(x, t) \mid -\alpha_1 r^2 \leq t < 0, x' \in K_r, \bar{x} \in S_{\beta r}\} \subseteq \text{supp}\phi,$$

- (4) $0 < \phi_0(z) < 1$, for $z \in \{(x, t) \mid -\alpha_1 r^2 \leq t \leq -\theta r^2, x' \in K_r, \bar{x} \in S_{\beta r}\}$.

Using Lemma 2.1 and the properties ϕ , we have the following lemma.

Lemma 3.3 (c.f. Lemma 3.2 in [25]) *Under the above notations, we have*

(a) *For $t < 0$, $|t|$ is small enough, then we have*

$$\langle \mathcal{C}^{-1}(|t|)e^{tB^T}x, e^{tB^T}x \rangle \approx |D_{|t|^{-\frac{1}{2}}}x|^2,$$

where C depends on B and λ .

(b) $Y\phi_0(z) \leq 0$, for $z \in \mathcal{Q}$.

Let $w = G(\frac{u}{h} + h^{\frac{1}{8}})$. Then we have the following Poincaré's type inequality.

Lemma 3.4 (Weak Poincaré inequality) *Let u be a non-negative weak solution of (1.7) in \mathcal{B}_1^- and $w = G(\frac{u}{h} + h^{\frac{1}{8}})$. Then there exists a constant $C = C(B, \lambda)$ such that for $r < \theta < 1$*

$$\begin{aligned} \int_{\mathcal{B}_{\theta r}^-} (w(z) - I_0)_+^2 &\leq C\theta^2 r^2 \int_{\mathcal{B}_{\frac{r}{\theta}}^-} |D_{m_0}w|^2 \\ &+ C(B, \lambda)h^{-\frac{9}{4}}|\frac{r}{\theta}|^{Q+2}|\frac{r}{\theta}|^{\frac{8}{Q+2}-\frac{4}{q}} \left(\|c\|_{L^q(\mathcal{B}_{\frac{r}{\theta}}^-)}^2 \|u\|_{L^\infty(\mathcal{B}_{\frac{r}{\theta}}^-)}^2 + \|f\|_{L^q(\mathcal{B}_{\frac{r}{\theta}}^-)}^2 \right), \end{aligned} \quad (3.6)$$

where I_0 is given by

$$I_0 = \max_{\mathcal{B}_{\theta r}^-} [I_1(z) + C_2(z)], \quad (3.7)$$

and

$$\begin{aligned} I_1(z) &= \int_{\mathcal{B}_{\frac{r}{\theta}}^-} [\langle \phi_1 A_0 D\phi_0, D\Gamma_1(z, \cdot) \rangle w - \Gamma_1(z, \cdot) w Y\phi](\zeta) d\zeta, \\ C_2(z) &= \int_{\mathcal{B}_{\frac{r}{\theta}}^-} [\langle \phi_0 A_0 D\phi_1, D\Gamma_1(z, \cdot) \rangle w](\zeta) d\zeta, \end{aligned} \quad (3.8)$$

where Γ_1 is the fundamental solution, and ϕ is given by (3.5).

Proof: We represent w in terms of the fundamental solution of Γ_1 . For $z \in \mathcal{B}_{\theta r}^-$, we have

$$\begin{aligned} w(z) &= \int_{\mathcal{B}_{\frac{r}{\theta}}^-} [\langle A_0 D(w\phi), D\Gamma_1(z, \cdot) \rangle - \Gamma_1(z, \cdot) Y(w\phi)](\zeta) d\zeta \\ &= I_1(z) + I_2(z) + I_3(z) + C_2(z), \end{aligned} \quad (3.9)$$

where $I_1(z)$ and $C_2(z)$ are given by (3.8) and

$$\begin{aligned} I_2(z) &= \int_{\mathcal{B}_{\frac{r}{\theta}}^-} [\langle (A_0 - A)Dw, D\Gamma_1(z, \cdot) \rangle \phi - \Gamma_1(z, \cdot) \langle ADw, D\phi \rangle](\zeta) d\zeta \\ &+ \int_{\mathcal{B}_{\frac{r}{\theta}}^-} [-\Gamma_1(z, \cdot) \phi b' \cdot D_{m_0}w + \Gamma_1(z, \cdot) \phi h^{-\frac{9}{8}}(|cu| + |f|)](\zeta) d\zeta, \end{aligned}$$

$$I_3(z) = \int_{\mathcal{B}_{\frac{r}{\theta}}^-} [\langle ADw, D(\Gamma_1(z, \cdot)\phi) \rangle - \Gamma_1(z, \cdot)\phi Yw + \Gamma_1(z, \cdot)\phi b' \cdot D_{m_0}w - \Gamma_1(z, \cdot)\phi \frac{|cu| + |f|}{h^{\frac{9}{8}}}] (\zeta) d\zeta.$$

From our assumption, w satisfies (3.1), and $\phi(\zeta)\Gamma_1(z, \cdot)$ is a test function of this semi-cylinder.

In fact, we let

$$\tilde{\chi}(\tau) = \begin{cases} 1 & \tau \leq 0, \\ 1 - n\tau & 0 \leq \tau \leq 1/n, \\ 0 & \tau \geq 1/n. \end{cases}$$

Then $\tilde{\chi}(\tau)\phi\Gamma_1(z, \cdot)$ can be a test function (see [2]). Let $n \rightarrow \infty$, we obtain $\phi\Gamma_1(z, \cdot)$ as a legitimate test function, and $I_3(z) \leq 0$. Then in $\mathcal{B}_{\theta r}^-$,

$$0 \leq (w(z) - I_0)_+ \leq I_2(z) = I_{21} + \cdots + I_{25}.$$

By Corollary 2.1, we have

$$\|I_{21}\|_{L^2(\mathcal{B}_{\theta r}^-)} \leq C(\lambda)\theta r \|I_{21}\|_{L^{2+\frac{4}{Q}}(\mathcal{B}_{\theta r}^-)} \leq C(B, \lambda)\theta r \|D_{m_0}w\|_{L^2(\mathcal{B}_{\frac{r}{\theta}}^-)}.$$

Similarly for I_{22} ,

$$\|I_{22}\|_{L^2(\mathcal{B}_{\theta r}^-)} \leq |\mathcal{B}_{\theta r}^-|^{\frac{1}{2} - \frac{Q-2}{2Q+4}} \|I_{22}\|_{L^{2k}(\mathcal{B}_{\theta r}^-)} \leq C(B, \lambda)\theta^2 r^2 \|D_{m_0}w D_{m_0}\phi\|_{L^2(\mathcal{B}_{\frac{r}{\theta}}^-)},$$

where $|D_{m_0}\phi| = |\phi_0 D_{m_0}\phi_1| = |\phi_0 \chi'(\theta|\xi')| \theta D_{m_0}(|\xi'|) \leq \frac{\theta}{r}$, thus

$$\|I_{22}\|_{L^2(\mathcal{B}_{\theta r}^-)} \leq C(B, \lambda)\theta^2 r \|D_{m_0}w\|_{L^2(\mathcal{B}_{\frac{r}{\theta}}^-)}.$$

For I_{23} , we have

$$\begin{aligned} \|I_{23}\|_{L^2(\mathcal{B}_{\theta r}^-)} &\leq |\mathcal{B}_{\theta r}^-|^{\frac{Q}{Q+2}} \|I_{23}\|_{L^{\frac{2Q+4}{Q}}(\mathcal{B}_{\theta r}^-)} \\ &\leq C(B, \lambda)\theta^2 r^2 \|b' \cdot D_{m_0}w\|_{L^{\frac{Q+4}{2Q+4}}(\mathcal{B}_{\frac{r}{\theta}}^-)} \\ &\leq C(B, \lambda)\theta^2 r^2 \|b'\|_{L^{Q+2}(\mathcal{B}_{\frac{r}{\theta}}^-)} \|D_{m_0}w\|_{L^2(\mathcal{B}_{\frac{r}{\theta}}^-)} \end{aligned}$$

For I_{24} , we have

$$\begin{aligned} \|I_{24}\|_{L^2(\mathcal{B}_{\theta r}^-)} &\leq C(B, \lambda)h^{-\frac{9}{8}} \|cu\|_{L^{\frac{2Q+4}{Q+6}}(\mathcal{B}_{\frac{r}{\theta}}^-)} \\ &\leq C(B, \lambda)h^{-\frac{9}{8}} \left|\frac{r}{\theta}\right|^{\frac{Q+2}{2}} \left|\frac{r}{\theta}\right|^{\frac{4}{Q+2} - \frac{2}{q}} \|c\|_{L^q(\mathcal{B}_{\frac{r}{\theta}}^-)} \|u\|_{L^\infty(\mathcal{B}_{\frac{r}{\theta}}^-)}. \end{aligned}$$

For I_{25} , we have

$$\begin{aligned} \|I_{25}\|_{L^2(\mathcal{B}_{\theta r}^-)} &\leq C(B, \lambda)h^{-\frac{9}{8}} \|f\|_{L^{\frac{2Q+4}{Q+6}}(\mathcal{B}_{\frac{r}{\theta}}^-)} \\ &\leq C(B, \lambda)h^{-\frac{9}{8}} \left|\frac{r}{\theta}\right|^{\frac{Q+2}{2}} \left|\frac{r}{\theta}\right|^{\frac{4}{Q+2} - \frac{2}{q}} \|f\|_{L^q(\mathcal{B}_{\frac{r}{\theta}}^-)}. \end{aligned}$$

Then we prove our lemma. \diamond

Next, we'll sketch the proof of the weak Sobolev inequality and L^∞ bounded estimates as in [4, 5]. In fact, we obtain two types of weak Sobolev equalities, where the representation formula of fundamental solution and potential estimates in Corollary 2.1 are used.

Lemma 3.5 (Sobolev estimate) *Under the assumptions $(\mathbf{H}_1 - \mathbf{H}_3)$, let u be a non-negative weak sub-solution of (1.7) in Ω .*

(i) *For $(x_0, t_0) \in \Omega$ and $\overline{\mathcal{B}_r^-(x_0, t_0)} \subset \Omega$, there holds*

$$\|\varphi u\|_{L^{2k}(\mathcal{B}_\rho^-(x_0, t_0))} \leq \frac{C}{r-\rho} (\|u\|_{L^2(\mathcal{B}_r^-(x_0, t_0))} + \|D_{m_0} u\|_{L^2(\mathcal{B}_r^-(x_0, t_0))}) + C \|f\|_{L^{\frac{2Q+4}{Q+4}}(\mathcal{B}_r^-(x_0, t_0))} \quad (3.10)$$

where φ be a cut-off function such that $\varphi = 1$ in \mathcal{B}_ρ^- , $\frac{1}{2} \leq \rho < r \leq 1$, $k = 1 + \frac{2}{Q}$, and C depends only on $q, N, \lambda, Q, \|b'\|_{L^{Q+2}(\mathcal{B}_r^-(x_0, t_0))}$ and $\|c\|_{L^q(\mathcal{B}_r^-(x_0, t_0))}$.

(ii) *Moreover, let $w = u^p$ with the positive integer $p > 1$, and we have the following similar estimate*

$$\begin{aligned} \|\varphi w\|_{L^{2k}(\mathcal{B}_\rho^-(x_0, t_0))} &\leq \frac{C}{r-\rho} (p^{\frac{1}{1-\beta}} \|w\|_{L^2(\mathcal{B}_r^-(x_0, t_0))} + \|D_{m_0} w\|_{L^2(\mathcal{B}_r^-(x_0, t_0))}) \\ &\quad + r^{p(2-\frac{Q+2}{q})+\frac{Q}{2}} \|f\|_{L^q(\mathcal{B}_r^-(x_0, t_0))}^p \end{aligned} \quad (3.11)$$

where $\beta = \frac{Q+2}{q} - 1 \in (0, 1)$ with $q > (Q+2)/2$.

Proof of Lemma 3.5: Step I: Test function. Similar as Lemma 3 in [5], $\Gamma_1(z, \cdot)\varphi$ can be a test function of (1.9), which is made by the cut-off at the singularity and dominated convergence theorem. For example, for the term $cu \in L^{\frac{2q}{q+2}}$ with $\frac{2q}{q+2} > 1$, since $q > \frac{Q+2}{2} \geq 2$, we have $\int_\Omega \Gamma_1(z, \zeta)\varphi(\zeta)c(\zeta)u(\zeta)d\zeta \in L^m$ with $\frac{1}{m} = \frac{1}{2} + \frac{1}{q} - \frac{2}{Q+2}$ due to Corollary 2.1, and obviously $m > 2$. Hence, we get

$$\int_\Omega \Gamma_1(z, \cdot)\varphi cu \chi\left(\frac{\|\zeta^{-1} \circ z\|}{\varepsilon}\right) \rightarrow \int_\Omega \Gamma_1(z, \cdot)\varphi cu,$$

for almost every $z \in R^{N+1}$, where χ is a smooth function satisfying $\chi(s) = 0$ for $s \in [0, 1]$ and $\chi(s) = 1$ for $s \geq 2$. For the others, we omitted it. Consequently, we get

$$\int_\Omega \Gamma_1(z, \cdot)\varphi Yu - (Du)^T AD(\varphi \Gamma_1(z, \cdot)) \geq \int_\Omega \Gamma_1(z, \cdot)\varphi(b' \cdot D_{m_0} u + cu + f). \quad (3.12)$$

Step II: Proof of (3.10). Let φ be a cut-off function such that $\varphi = 1$ in \mathcal{B}_ρ^- and $\varphi = 0$ outside of \mathcal{B}_r^- ; furthermore, $|\partial_t \varphi| + |D\varphi| \leq \frac{C}{r-\rho}$. We represent u in terms of the fundamental

solution of Γ_1 . For $z \in \mathcal{B}_\rho^-$, we have

$$\begin{aligned} u(z)\varphi(z) &= \int_{\mathcal{B}_r^-} [\langle A_0 D(u\varphi), D\Gamma_1(z, \cdot) \rangle - \Gamma_1(z, \cdot) Y(u\varphi)](\zeta) d\zeta \\ &= I_1(z) + I_2(z) + I_3(z) + I_4(z), \end{aligned}$$

where $I_1(z) - I_4(z)$ are as follows:

$$I_1(z) = \int_{\mathcal{B}_r^-} [\langle A_0 D\varphi, D\Gamma_1(z, \cdot) \rangle u - \Gamma_1(z, \cdot) u Y\varphi](\zeta) d\zeta,$$

$$I_2(z) = \int_{\mathcal{B}_r^-} [\langle (A_0 - A)Du, D\Gamma_1(z, \cdot) \rangle \varphi - \Gamma_1(z, \cdot) \langle ADu, D\varphi \rangle](\zeta) d\zeta,$$

$$I_3(z) = \int_{\mathcal{B}_r^-} [\langle ADu, D(\Gamma_1(z, \cdot)\varphi) \rangle - \Gamma_1(z, \cdot) \varphi Y u + \Gamma_1(z, \cdot) \varphi (b' \cdot D_{m_0} u + cu + f)](\zeta) d\zeta'$$

and

$$I_4(z) = - \int_{\mathcal{B}_r^-} [\Gamma_1(z, \cdot) \varphi (b' \cdot D_{m_0} u + cu + f)](\zeta)(\zeta) d\zeta = I_{41}(z) + I_{42}(z) + I_{43}(z),$$

Obviously, by (3.12) we have $I_3(z) \leq 0$. For the term I_1 , by Corollary 2.1 and Hölder inequality we have

$$\|I_1\|_{L^{2k}(\mathcal{B}_r^-)} \leq C \|u D_{m_0} \varphi\|_{L^2(R^{N+1})} + C |\mathcal{B}_r^-|^{\frac{2}{Q}} \|u Y \varphi\|_{L^2(R^{N+1})} \leq \frac{C}{r - \rho} \|u\|_{L^2(\mathcal{B}_r^-)}$$

Similarly, for I_2 we have

$$\|I_2\|_{L^{2k}(\mathcal{B}_r^-)} \leq \frac{C}{r - \rho} \|D_{m_0} u\|_{L^2(\mathcal{B}_r^-)}$$

Next, we estimate I_4 ,

$$\|I_{41}\|_{L^{2k}(\mathcal{B}_r^-)} \leq C \|\varphi b' \cdot D_{m_0} u\|_{L^{\frac{2Q+4}{Q+4}}(R^{N+1})} \leq C \|b'\|_{L^{Q+2}(\mathcal{B}_r^-)} \|D_{m_0} u\|_{L^2(\mathcal{B}_r^-)},$$

$$\|I_{43}\|_{L^{2k}(\mathcal{B}_r^-)} \leq C \|\varphi f\|_{L^{\frac{2Q+4}{Q+4}}(R^{N+1})} \leq C \|f\|_{L^{\frac{2Q+4}{Q+4}}(\mathcal{B}_r^-)},$$

and

$$\begin{aligned} \|I_{42}\|_{L^{2k}(\mathcal{B}_r^-)} &\leq C \|c|\varphi u|^\beta |\varphi u|^{1-\beta}\|_{L^{\frac{2Q+4}{Q+4}}(R^{N+1})} \leq C \|c\|_{L^q(\mathcal{B}_r^-)} \|\varphi u\|_{L^{2k}(\mathcal{B}_r^-)}^\beta \|u\varphi\|_{L^2(\mathcal{B}_r^-)}^{1-\beta} \\ &\leq \frac{1}{2} \|\varphi u\|_{L^{2k}(\mathcal{B}_r^-)} + C \|c\|_{L^q(\mathcal{B}_r^-)}^{\frac{1}{1-\beta}} \|u\varphi\|_{L^2(\mathcal{B}_r^-)}, \end{aligned}$$

where $\beta = \frac{Q+2}{q} - 1 \in (0, 1)$.

Concluding the above estimates, we have

$$\begin{aligned} \|u\varphi\|_{L^{2k}} &\leq \frac{C}{r-\rho} \|u\|_{L^2(\mathcal{B}_r^-)} + C\left(\frac{1}{r-\rho} + \|b'\|_{L^{Q+2}(\mathcal{B}_r^-)}\right) \|D_{m_0}u\|_{L^2(\mathcal{B}_r^-)} \\ &\quad + C\|f\|_{L^{\frac{2Q+4}{Q+4}}(\mathcal{B}_r^-)} + C\|c\|_{L^q(\mathcal{B}_r^-)}^{\frac{q}{2q-Q-2}} \|u\varphi\|_{L^2(\mathcal{B}_r^-)}. \end{aligned}$$

Then we complete the proof of (3.10).

Step III: Proof of (3.11). To prove the inequality (3.11), if $\|w\|_{L^2(\mathcal{B}_r^-(x_0, t_0))} + \|D_{m_0}w\|_{L^2(\mathcal{B}_r^-(x_0, t_0))} < \infty$, firstly we have $pu^{p-1}\Gamma_1(z, \cdot)\varphi$ as a test function of (1.7) and there holds

$$\int_{\Omega} \Gamma_1(z, \cdot)\varphi Yw - (Dw)^T AD(\varphi\Gamma_1(z, \cdot)) \geq \int_{\Omega} \Gamma_1(z, \cdot)\varphi(b' \cdot D_{m_0}w + pcw + pu^{p-1}f). \quad (3.13)$$

Next, we deal with the terms pcu^p and $pu^{p-1}|f|$ only, and other terms are similar as Step II. Write the last two terms of the righthand of (3.13) as I'_{42} and I'_{43} .

Case I: $f \in L^q$ and $p > \frac{q(Q+2)}{2(Q+2-q)}$. At this moment, we have

$$\|I'_{42}\|_{L^{2k}(\mathcal{B}_r^-)} \leq \frac{1}{2} \|\varphi w\|_{L^{2k}(\mathcal{B}_r^-)} + Cp^{\frac{1}{1-\beta}} \|c\|_{L^q(\mathcal{B}_r^-)}^{\frac{1}{1-\beta}} \|w\varphi\|_{L^2(\mathcal{B}_r^-)},$$

where $\beta = \frac{Q+2}{q} - 1 \in (0, 1)$. And

$$\begin{aligned} \|I'_{43}\|_{L^{2k}(\mathcal{B}_r^-)} &\leq C\|\varphi pu^{p-1}f\|_{L^{\frac{2Q+4}{Q+4}}(R^{N+1})} \leq Cp\|f\|_{L^q(\mathcal{B}_r^-)} \|\varphi w\|_{L^{2k}(\mathcal{B}_r^-)}^{\beta'} \|w\varphi\|_{L^2(\mathcal{B}_r^-)}^{1-\frac{1}{p}-\beta'} \\ &\leq \frac{1}{2} \|\varphi w\|_{L^{2k}(\mathcal{B}_r^-)} + \frac{C}{r-\rho} p^{\frac{1}{1-\frac{1}{p}-\beta'}} \|w\varphi\|_{L^2(\mathcal{B}_r^-)} + (r-\rho)^{p(1-\frac{1}{p}-\beta')} \|f\|_{L^q(\mathcal{B}_r^-)}^p \end{aligned}$$

where β' satisfies

$$\frac{1}{q} + \frac{\beta'}{2k} + \frac{1-\frac{1}{p}-\beta'}{2} = \frac{Q+4}{2Q+4}, \quad k = 1 + \frac{2}{Q}$$

and hence $\beta' = (\frac{1}{q} - \frac{1}{2p})(Q+2) - 1 \in (0, 1)$ with $q > (Q+2)/2$.

It is easy to check that

$$\frac{1}{1-\beta} > \frac{1}{1-\frac{1}{p}-\beta'},$$

which yields that

$$\begin{aligned} \|w\varphi\|_{L^{2k}(\mathcal{B}_r^-)} &\leq \frac{C}{r-\rho} p^{\frac{1}{1-\beta}} \|w\|_{L^2(\mathcal{B}_r^-)} + \frac{C}{r-\rho} \|D_{m_0}w\|_{L^2(\mathcal{B}_r^-)} \\ &\quad + r^{p(2-\frac{Q+2}{q})+\frac{Q}{2}} \|f\|_{L^q(\mathcal{B}_r^-)}^p \end{aligned}$$

Case II: $f \in L^q$ and $1 < p \leq \frac{q(Q+2)}{2(Q+2-q)}$. Now, we have

$$\begin{aligned} \|I'_{43}\|_{L^{2k}(\mathcal{B}_r^-)} &\leq C \|\varphi p u^{p-1} f\|_{L^{\frac{2Q+4}{Q+4}}(R^{N+1})} \leq C p \|f\|_{L^q(\mathcal{B}_r^-)} \|w\varphi\|_{L^2(\mathcal{B}_r^-)}^{\frac{p-1}{p}} r^{(Q+2)(\frac{Q+4}{2Q+4}-\frac{1}{q}-\frac{p-1}{2p})} \\ &\leq \frac{C}{r-\rho} p^{\frac{p-1}{p}} \|w\varphi\|_{L^2(\mathcal{B}_r^-)} + r^{p(Q+2)(\frac{2}{Q+2}-\frac{1}{q}+\frac{1}{2p})-1} \|f\|_{L^q(\mathcal{B}_r^-)}^p \end{aligned}$$

Then

$$\begin{aligned} \|w\varphi\|_{L^{2k}} &\leq \frac{C}{r-\rho} p^{\frac{1}{1-\beta}} \|w\|_{L^2(\mathcal{B}_r^-)} + \frac{C}{r-\rho} \|D_{m_0} w\|_{L^2(\mathcal{B}_r^-)} \\ &\quad + r^{p(2-\frac{Q+2}{q})+\frac{Q}{2}} \|f\|_{L^q(\mathcal{B}_r^-)}^p. \end{aligned}$$

Hence, we can complete the proof of (3.11). \diamond

We also obtained the bounded property of nonnegative weak sub-solution of (1.7) similarly as Cinti, Pascucci and Polidoro[5] by using the Moser's iterative method, which states as follows.

Lemma 3.6 (L^∞ estimate) *Under the assumptions $(\mathbf{H}_1 - \mathbf{H}_3)$, let u be a non-negative weak sub-solution of (1.7) in Ω . Let $(x_0, t_0) \in \Omega$ and $\overline{\mathcal{B}_r^-(x_0, t_0)} \subset \Omega$ and $p \geq 1$. Then there exists a positive constant C which depends only on $q, N, \lambda, Q, \|b'\|_{L^{Q+2}(\mathcal{B}_r^-(x_0, t_0))}, \|c\|_{L^q(\mathcal{B}_r^-(x_0, t_0))}$ and $\|f\|_{L^q(\mathcal{B}_r^-(x_0, t_0))}$ such that, for $0 < r \leq 1$*

$$\sup_{\mathcal{B}_{\frac{r}{2}}^-(x_0, t_0)} u^p \leq \frac{C}{r^{Q+2}} \int_{\mathcal{B}_r^-(x_0, t_0)} u^p + C, \quad q > \frac{Q+2}{2}$$

provided that the last integral converges.

Proof. Since u is a non-negative weak sub-solution of (1.7), we have for any nonnegative $\varphi \in C_0^\infty(\Omega)$, there holds

$$\int_{\Omega} \varphi Y u - (Du)^T A D \varphi \geq \int_{\Omega} \varphi (b' \cdot D_{m_0} u + cu + f).$$

Without loss of generality, we assume that $\Omega = \mathcal{B}_1^-$. Taking $\varphi = \eta(\|x\|) p u^{2p-1}$ and $w = u^p$, where $\eta(x) = 1$ if $\|x\| \leq r$ and $\eta(x) = 0$ if $\|x\| \geq 2r$, then we have

$$\begin{aligned} &\int_{t_0}^{t_1} \int \left[\frac{1}{2} \eta^2 \partial_t (w^2) + \eta^2 (Dw)^T A D w + 2\eta w (Dw)^T A D \eta \right] dx dt \\ &\leq - \int_{t_0}^{t_1} \int \left[\frac{1}{2} w^2 \langle x, BD \rangle \eta^2 + \eta^2 w b' \cdot D_{m_0} w + c p w^2 \eta^2 + p f w u^{p-1} \eta^2 \right] \\ &\leq - \int_{t_0}^{t_1} \int \frac{1}{2} w^2 \langle x, BD \rangle \eta^2 + I_1 + I_2 + I_3 \end{aligned}$$

and $t_0 \in (-2r^2, -r^2)$ such that

$$\int \eta^2 w^2(x, t_0) dx \leq \frac{1}{r^2} \int_{-2r^2}^{-r^2} \int \eta^2 w^2(x, t) dx dt.$$

Moreover, by scaling

$$| \langle x, BD \rangle \eta | \leq \frac{C}{r^2}, \quad |D_{m_0} \eta| \leq \frac{C}{r},$$

then

$$I_1 \leq \frac{1}{2} \int_{t_0}^{t_1} \int \eta^2 (Dw)^T ADw + C \|b'\|_{L^s(\mathcal{B}_{2r}^-)}^{\frac{2}{s}} \|\eta w\|_{L^{2k}(\mathcal{B}_1^-)}^2$$

where $\frac{1}{k} + \frac{2}{s} = 1$ and $s = Q + 2$.

$$I_2 \leq p \|c\|_{L^q(\mathcal{B}_{2r}^-)}^2 \|\eta w\|_{L^{2q'}(\mathcal{B}_1^-)}^2,$$

and

$$\begin{aligned} I_3 &\leq p \|f\|_{L^q(\mathcal{B}_{2r}^-)}^2 \|\eta w\|_{L^{2q'}(\mathcal{B}_1^-)}^{1-\frac{1}{2p}} r^{\frac{Q+2}{2pq'}} \\ &\leq \|f\|_{L^q(\mathcal{B}_{2r}^-)}^2 (p^3 \|\eta w\|_{L^{2q'}(\mathcal{B}_1^-)}^2 + p^{-2p} r^{\frac{Q+2}{q'}}) \end{aligned}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$ and obviously $q' < k$.

Let $c_1(r) = \|c\|_{L^q(\mathcal{B}_r^-)}$ and $\gamma = 1 - \frac{Q+2}{2q} \in (0, 1)$, then $\frac{Q+2}{q'} = Q + 2\gamma$. Concluding the above estimates, for $t \in (-r^2, 0)$, we get

$$\begin{aligned} &\int \eta^2 w^2(x, t) dx + \int_{-r^2}^0 \int \eta^2 (Dw)^T ADw dx dt \leq \frac{C}{r^2} \int \int \eta^2 w^2(x, t) dx dt \\ &+ C c_1(2r) r^{\frac{Q+2}{q'}} p^3 \left(r^{-Q-2} \int_{-r^2}^0 \int |\eta w|^{2q'} dx dt \right)^{\frac{1}{q'}} + C \|f\|_{L^q(\mathcal{B}_{2r}^-)}^2 r^{\frac{Q+2}{q'}} p^{-2p} \end{aligned}$$

Using the embedding inequality (3.11), we have

$$\begin{aligned} &\left(r^{-Q-2} \int_{-r^2}^0 \int |\eta w|^{2k} dx dt \right)^{\frac{1}{k}} \leq \frac{C}{r^{Q+2}} p^{\frac{2}{1-\beta}} \int \int \eta^2 w^2(x, t) dx dt \\ &+ C c_1(2r) r^{2\gamma} p^3 \left(r^{-Q-2} \int_{-r^2}^0 \int |\eta w|^{2q'} dx dt \right)^{\frac{1}{q'}} \\ &+ C r^{2\gamma} p^{-2p} \|f\|_{L^q(\mathcal{B}_{2r}^-)}^2 + r^{2p(2-\frac{Q+2}{q})} \|f\|_{L^q(\mathcal{B}_{2r}^-)}^{2p} \end{aligned}$$

where $\beta = \frac{Q+2}{q} - 1 \in (0, 1)$.

Then the standard iterative technique yields the required result due to $q' < k$ (see, also P531 in [14]). \diamond

Now we apply Lemma 3.4 to the function

$$w = G\left(\frac{u}{h} + h^{\frac{1}{8}}\right).$$

If u is a weak solution of (1.7), obviously w is an almost weak sub-solution as in (3.1). We estimate the value of I_0 given by (3.7) and (3.8) in Lemma 3.4.

Lemma 3.7 (Lemma 3.4 in [25]) *Under the assumptions of Lemma 3.4, there exist constants λ_0 , r_0 and $r_0 < \theta$. λ_0 only depends on constants α , β , λ , B , N , and φ , $0 < \lambda_0 < 1$, such that for $r < r_0$*

$$|I_0| \leq \lambda_0 \ln(h^{-\frac{1}{8}}).$$

Lemma 3.8 *Suppose that $u(x, t) \geq 0$ be a solution of equation (1.7) in \mathcal{B}_r^- centered at $(0, 0)$ and*

$$\text{mes}\{(x, t) \in \mathcal{B}_r^-, \quad u \geq 1\} \geq \frac{1}{2} \text{mes}(\mathcal{B}_r^-).$$

Then there exist constant θ and h_0 , $0 < \theta, h_0 < 1$ which only depend on B , λ , λ_0 and N such that

$$u(x, t) \geq h_0 \quad \text{in} \quad \mathcal{B}_{\theta r}^-.$$

Proof: We consider $w = G(\frac{u}{h} + h^{\frac{1}{8}})$ for $0 < h < \frac{1}{4}$, to be decided. Take $r = \theta h^{\frac{q(Q+2)}{2q-Q-2}}$, by applying Lemma 3.4 to w , and we have

$$\int_{\mathcal{B}_{\theta r}^-} (w - I_0)_+^2 \leq C(B, \lambda) \frac{\theta r^2}{|\mathcal{B}_{\theta r}^-|} \int_{\mathcal{B}_r^-} |D_{m_0} w|^2 + C(\theta, B, \lambda, \|c\|_{L^q}, \|u\|_{L^2}, \|f\|_{L^q}) h.$$

Let $\tilde{u} = \frac{u}{h}$, then \tilde{u} satisfies the conditions of Lemma 3.2. We can get similar estimates as (3.2)-(3.4), hence we have

$$\begin{aligned} & C(B, \lambda) \frac{\theta r^2}{|\mathcal{B}_{\theta r}^-|} \int_{\mathcal{B}_r^-} |D_{m_0} w|^2 \\ & \leq C(B, \lambda) \frac{\theta r^2}{|\mathcal{B}_{\theta r}^-|} [C(B, \lambda)(1 - \beta)^{-2} \beta^{-Q} + \frac{4}{5} \ln(h^{-\frac{1}{8}})] \text{mes}(K_{\beta r} \times S_{\beta r}) \\ & \leq C(\theta, B, \lambda) \ln(h^{-\frac{1}{8}}), \end{aligned}$$

where θ has been chosen. By Lemma 3.6, there exists a constant, still denoted by θ , such that for $z \in \mathcal{B}_{\theta r}^-$,

$$w - I_0 \leq C(B, \lambda) (\ln(h^{-\frac{1}{8}}))^{\frac{1}{2}}. \quad (3.14)$$

Therefore we may choose h_0 small enough, so that

$$C(\ln(\frac{1}{h_0^{\frac{1}{8}}}))^{\frac{1}{2}} \leq \ln(\frac{1}{2h_0^{\frac{1}{8}}}) - \lambda_0 \ln(\frac{1}{h_0^{\frac{1}{8}}}).$$

Then Lemma 3.7 and (3.14) implies

$$\max_{\mathcal{B}_{\theta_r}^-} \frac{h_0}{u + h_0^{\frac{9}{8}}} \leq \frac{1}{2h_0^{\frac{1}{8}}},$$

which implies $\min_{\mathcal{B}_{\theta_r}^-} u \geq h_0^{\frac{9}{8}}$, then we finished the proof of this lemma. \diamond

4 Proof of our main results

Proof of Theorem 1.1. This is similar to that in [25]. We may assume that $M = \max_{\mathcal{B}_r^-} (+u) = \max_{\mathcal{B}_r^-} (-u)$, otherwise we replace u by $u - C$, since u is bounded locally. Then either $1 + \frac{u}{M}$ or $1 - \frac{u}{M}$ satisfies the assumption of Lemma 3.8, and we suppose $1 + \frac{u}{M}$ does it, thus Lemma 3.8 implies existing $h_0 > 0$ such that $\inf_{\mathcal{B}_{\theta_r}^-} (1 + \frac{u}{M}) \geq h_0$, i.e. $u \geq M(h_0 - 1)$, then

$$Osc_{\mathcal{B}_{\theta_r}^-} u \leq M - M(h_0 - 1) \leq (1 - \frac{h_0}{2}) Osc_{\mathcal{B}_r^-} u,$$

which implies the C^α regularity of u near point $(0,0)$ by the standard iteration arguments. By the left invariant translation group action, we know that u is C^α in the interior. \diamond

Proof of Corollary 1.1. We make the following transform:

$$\tilde{\tau} = \sqrt{U}\tau, \quad \tilde{\xi} = \xi, \quad \tilde{\eta} = \sqrt{U}\eta,$$

then we have

$$\begin{aligned} \partial_\tau w^{-1} &= \sqrt{U} \partial_{\tilde{\tau}} w^{-1} + (\frac{u_t}{\sqrt{U}} - \frac{U_t}{2\sqrt{U}} \eta) \partial_{\tilde{\eta}} w^{-1}, \\ \sqrt{U} \tilde{\eta} \partial_{\tilde{\xi}} (w^{-1}) &= \sqrt{U} \tilde{\eta} \partial_{\tilde{\xi}} w^{-1}, \\ A \partial_{\tilde{\eta}} (w^{-1}) &= A \sqrt{U} \partial_{\tilde{\eta}} w^{-1}, \\ -\partial_{\tilde{\eta}} w &= -\sqrt{U} \partial_{\tilde{\eta}} (w^2 \sqrt{U} \partial_{\tilde{\eta}} w^{-1}) \end{aligned}$$

Hence, the equation (1.5) can be reduced into

$$\partial_{\tilde{\tau}} w^{-1} + \tilde{\eta} \partial_{\tilde{\xi}} w^{-1} + \tilde{A} \partial_{\tilde{\eta}} w^{-1} - \frac{\tilde{B}}{\sqrt{U}} w^{-1} = -\partial_{\tilde{\eta}} (w^2 \sqrt{U} \partial_{\tilde{\eta}} w^{-1}) \quad (4.1)$$

where $\tilde{A} = (1 - \eta^2) \partial_x U + (1 - \frac{3}{2} \eta) \frac{U_t}{U} + \frac{u_t}{U}$ and $\tilde{B} = \eta \partial_x U + \frac{U_t}{U}$.

Considering the space dimension 2 and at this time $Q = 4$, by the assumptions and embedding inequality we get $0 < U \in C_{loc}^0$ and

$$\begin{aligned} b' &= \tilde{A} = (1 - \eta^2) \partial_x U + (1 - \frac{3}{2} \eta) \frac{U_t}{U} + \frac{u_t}{U} \in L^{Q+2}(= L^6), \\ c &= \frac{\tilde{B}}{\sqrt{U}} \in L^6, \end{aligned}$$

which satisfy the conditions of Theorem 1.1. Thus w^{-1} and $\partial_y u$ is Hölder continuous in the interior of the domain. On the other hand, $\partial_y u \in L_{loc}^\infty$, and the interpolation inequality yields that u is Hölder continuous. The proof is complete. \diamond

5 Appendix: G-function

Next, we introduce some properties of G-function, which was mentioned in [14] (see also [10]). Here, we give a detailed description for completeness.

Lemma 5.1 (G-function) *There exists a function $G(t) : (0, +\infty) \rightarrow \mathbb{R}$ such that*

$$\left\{ \begin{array}{ll} i) & G''(t) \geq G'(t)^2, \quad t > 0; \\ ii) & G(u) = 0, \quad t \geq 1; \\ iii) & G'(u) \sim -\ln t, \quad t \rightarrow 0^+; \\ iv) & 0 \leq -G'(t) \leq \frac{1}{t}, \quad 0 < t \leq \frac{1}{4}. \end{array} \right.$$

Proof: Let $h_0(t)$ be a simple function as follows:

$$h_0(t) = \begin{cases} -1, & t \leq 1 \\ 0, & t > 1. \end{cases}$$

By standard mollifying technique, one can obtain a smooth function $h(t) \in C^\infty(\mathbb{R})$

$$\left\{ \begin{array}{ll} i) & h(t) = h_0(t), \quad t \leq \frac{1}{2} \text{ or } t > 2; \\ ii) & h'(t) \geq 0, \quad t \geq 0; \\ iii) & h(t) \leq 0, \quad t \geq 0; \\ iv) & \int_0^2 h(t) dt = -1. \end{array} \right.$$

Again, we let $f(t) = \int_0^t h(t)dt$, then

$$\begin{cases} i) & f'(t) = h(t) \leq 0, \quad t \geq 0; \\ ii) & -t \leq f(t) < 0, \quad t > 0; \\ iii) & f(0) = 0, \quad f(t) = -1, \quad t \geq 2. \end{cases}$$

Next, write $g(t) = -\ln(-f(t))$, then we have

$$g'(t) = -\frac{f'(t)}{f(t)} = -\frac{h(t)}{f(t)} \leq 0,$$

and

$$g''(t) = \frac{f'(t)^2 - f(t)h'(t)}{f(t)^2} \geq \frac{f'(t)^2}{f(t)^2} = g'(t)^2.$$

Moreover, we have

$$\lim_{t \rightarrow 0^+} \frac{g(t)}{-\ln t} = \lim_{t \rightarrow 0^+} \frac{g'(t)}{-t^{-1}} = \lim_{t \rightarrow 0^+} \frac{th(t)}{f(t)} = 1,$$

and

$$g(t) = -\ln(-\int_0^t h(t)dt) = 0, \quad t \geq 2.$$

Hence, the proof of (i)-(iii) is complete by choosing $G(t) = g(2t)$. Finally, we come to prove (iv). Since the function $\tilde{g}(t) = g(\mu t + \nu)$ for any $\mu, \nu > 0$ satisfies

$$\tilde{g}'(t) \leq 0, \quad \tilde{g}''(t) \geq 0, \quad \text{for } t \geq 0,$$

which implies both $|\tilde{g}'(t)| = -\tilde{g}'(t)$ and $\tilde{g}(t)$ attain its maximum at $t = 0$ when $t \geq 0$. Then

$$|\tilde{g}'(t)| \leq -\tilde{g}'(0) = -\mu g'(\nu).$$

Note that $h(= -1)$ for $t \leq \frac{1}{2}$, and we have $-f(t) + th(t) = 0$ when $0 \leq t \leq \frac{1}{2}$. Then for $0 < t \leq \frac{1}{2}$, we get

$$\begin{aligned} [-tg'(t)]' &= -g'(t) - tg''(t) \\ &= \frac{h(t)}{f(t)} - t \frac{f'(t)^2 - f(t)h'(t)}{f(t)^2} \\ &\leq \frac{-h(t)(-f(t) + th(t))}{f(t)^2} = 0, \end{aligned}$$

and

$$|-tg'(t)| \leq \lim_{t \rightarrow 0^+} |-tg'(t)| = 1,$$

which yields that

$$|g'(t)| \leq \frac{1}{t}, \quad 0 < t \leq \frac{1}{2}.$$

Thus, the proof is complete. \diamond

Acknowledgments. W. Wang was supported by NSFC 11301048, 11671067 and part of The Institute of Mathematical Sciences of Chinese University of Hong Kong. L. Zhang was partially supported by NSFC under grant 11471320 and 11631008.

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